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Simultaneous amplification and attenuation in isotropic chiral materials

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Abstract

The electromagnetic field phasors in an isotropic chiral material (ICM) are superpositions of two Beltrami fields of different handedness. Application of the Bruggeman homogenization formalism to two-component composite materials delivers ICMs wherein Beltrami fields of one handedness attenuate whereas Beltrami fields of the other handedness amplify. One component material is a dissipative ICM, the other an active dielectric material. The range of the volume fraction of the active component material for which simultaneous amplification and attenuation is exhibited decreases—but does not vanish—as the ICM component becomes more dissipative and as its chirality parameter reduces in magnitude.

Keywords: attenuation, amplification, Beltrami field, Bruggeman homogenization formalism, left-circular polarization, right-circular polarization

(Some figures may appear in colour only in the online journal)

1. Introduction

The issues of gain and loss are currently prominent ones in electromagnetics, as active component materials are being introduced in electromagnetic metamaterials in order to overcome losses [1–3]. Depending on the imaginary part of its permittivity scalar $\varepsilon(\omega)$ at angular frequency ω , an isotropic dielectric material is either (i) dissipative if $\text{Im}[\varepsilon(\omega)] > 0$ or (ii) active if $\text{Im}[\varepsilon(\omega)] < 0$ or (iii) neither if $\text{Im}[\varepsilon(\omega)] = 0$, provided that the electromagnetic fields are assumed to depend as $\exp(-i\omega t)$ on time t [4]. Whether an anisotropic dielectric material is dissipative or active is determined by the imaginary part of its permittivity dyadic $\underline{\underline{\varepsilon}}(\omega)$ [5]. Furthermore, anisotropic dielectric materials for which the imaginary part of $\underline{\underline{\varepsilon}}(\omega)$

is indefinite can exhibit gain for certain propagation directions and loss for other propagation directions [6].

This article concerns the issue of simultaneous attenuation and amplification during plane wave propagation along any specific direction in an isotropic chiral material (ICM). Examples of such materials abound in nature, e.g., organic substances whose molecules contain mirror-asymmetric arrangements of atoms [7–9]. Also, artificial ICMs can be engineered [10–12]. Very notably, during the last ten years chiral metamaterials have been the subject of vigorous research directed towards the manifestation of negative refraction [13–15].

2. Plane wave propagation in ICMs

An ICM is characterized by the frequency-domain Tellegen constitutive relations [16]⁴

$$\left. \begin{aligned} \mathbf{D}(\mathbf{r}) &= \varepsilon \mathbf{E}(\mathbf{r}) + i\xi \mathbf{H}(\mathbf{r}) \\ \mathbf{B}(\mathbf{r}) &= -i\xi \mathbf{E}(\mathbf{r}) + \mu \mathbf{H}(\mathbf{r}) \end{aligned} \right\} \quad (1)$$

⁴ Here, and henceforth, the dependencies of the constitutive parameters and field phasors on ω are not explicitly displayed.

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The permittivity scalar ε , the permeability scalar μ , and the chirality pseudoscalar ξ are frequency dependent and complex valued, in consequence of the principle of causality enshrined as the Kramers–Kronig relations [17]. The Bohren decomposition [18]

$$\left. \begin{aligned} \mathbf{E}(\mathbf{r}) &= \mathbf{Q}_L(\mathbf{r}) - i\eta\mathbf{Q}_R(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}) &= (i\eta)^{-1}\mathbf{Q}_L(\mathbf{r}) + \mathbf{Q}_R(\mathbf{r}) \end{aligned} \right\} \quad (2)$$

is employed to represent \mathbf{E} and \mathbf{H} as superpositions of a left-handed Beltrami field \mathbf{Q}_L and a right-handed Beltrami field \mathbf{Q}_R , with $\eta = \mu^{1/2}\varepsilon^{-1/2}$ as the intrinsic impedance [16]. In source-free regions, the two Beltrami fields obey the relations

$$\left. \begin{aligned} \nabla \times \mathbf{Q}_L(\mathbf{r}) &= k_L \mathbf{Q}_L(\mathbf{r}) \\ \nabla \times \mathbf{Q}_R(\mathbf{r}) &= -k_R \mathbf{Q}_R(\mathbf{r}) \end{aligned} \right\}, \quad (3)$$

where the wavenumbers

$$\left. \begin{aligned} k_L &= \omega(\mu^{1/2}\varepsilon^{1/2} + \xi) \\ k_R &= \omega(\mu^{1/2}\varepsilon^{1/2} - \xi) \end{aligned} \right\}. \quad (4)$$

Let us consider plane wave propagation along the $+z$ direction. Then $\mathbf{Q}_L = (\hat{\mathbf{u}}_x + i\hat{\mathbf{u}}_y)\exp(ik_L z)$ is a left-circularly polarized (LCP) plane wave and $\mathbf{Q}_R = (\hat{\mathbf{u}}_x - i\hat{\mathbf{u}}_y)\exp(ik_R z)$ is a right-circularly polarized (RCP) plane wave. Could the LCP plane wave lose energy and the RCP plane wave gain energy, or vice versa, as z increases? More generally, could Beltrami waves of one handedness attenuate while Beltrami waves of the other handedness amplify, even if these Beltrami waves are not plane waves, but, say, spherical or cylindrical waves? In other words, could $\text{Im}(k_L)$ and $\text{Im}(k_R)$ be of opposite signs, but $\text{Re}(k_L)$ and $\text{Re}(k_R)$ have the same signs? If yes, then ICM research is promising for circular polarizers of a new type.

3. Homogenization

A perusal of literature on ICMs did not turn up any example for which $\text{Im}(k_L)\text{Im}(k_R) < 0$ but $\text{Re}(k_L)\text{Re}(k_R) > 0$. Hence, we decided to investigate the effective electromagnetic properties of a particulate composite material comprising an active dielectric material and a dissipative ICM. If the component materials can be regarded as being randomly distributed as electrically small particles, then the composite material could be homogenized into an ICM itself [19].

3.1. Simultaneous amplification and attenuation

Let the component material labeled ‘a’ be an active isotropic dielectric material specified by the permittivity ε^a such that $\text{Re}(\varepsilon^a) > 0$ and $\text{Im}(\varepsilon^a) < 0$. Let the component material ‘b’ be a dissipative ICM, characterized by constitutive relations of the form given in equation (1), but with the superscript ‘b’ attached to the constitutive parameters ε , ξ , and μ therein. We used the well-established Bruggeman formalism [20] to estimate the constitutive parameters ε^{Br} , ξ^{Br} , and μ^{Br} of the homogenized composite material (HCM), per equation (1) but

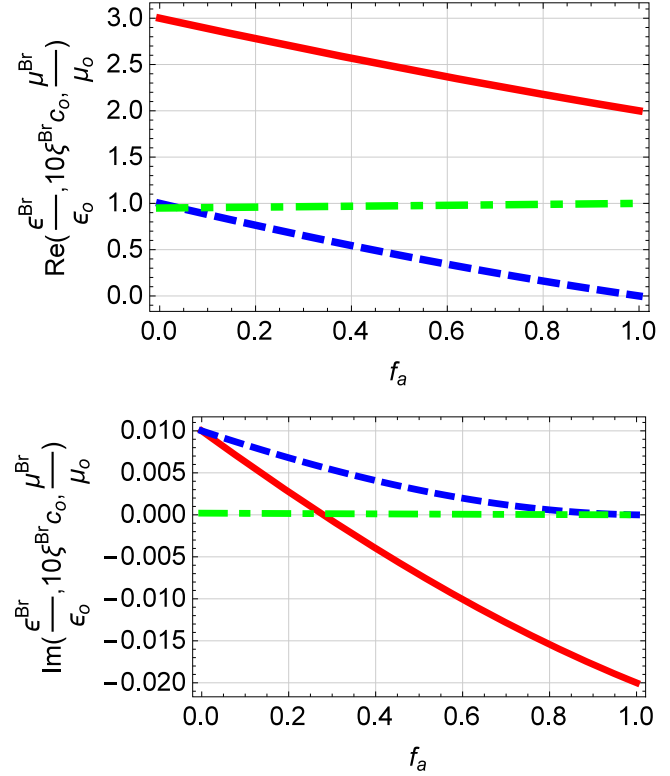


Figure 1. The real and imaginary parts of the constitutive parameters $\varepsilon^{\text{Br}}/\varepsilon_0$ (red solid curves), $\xi^{\text{Br}}c_0$ (blue dashed curves), and μ^{Br}/μ_0 (green broken-dashed curves) of the HCM, as estimated using the Bruggeman formalism, plotted against volume fraction f_a . The real and imaginary parts of $\xi^{\text{Br}}c_0$ are scaled by a factor of 10. See the text for values of ε^a , ε^b , ξ^b , and μ^b .

with the superscript ‘Br’ attached to the constitutive parameters ε , ξ , and μ therein. Let $f_a \in [0, 1]$ denote the volume fraction of component material ‘a’, the volume fraction of component material ‘b’ being $f_b = 1 - f_a$.

Significantly, for the ICM component we set $\text{Re}(\varepsilon^b) > 0$, thereby avoiding the condition $\text{Re}(\varepsilon^a)\text{Re}(\varepsilon^b) < 0$ that is likely to give rise to physically implausible results (by extrapolation from the corresponding homogenization scenario for isotropic dielectric HCMs [21, 22]). This is not at all a limitation. For examples, the isotropic chiral metamaterials of Kwon *et al* [14] has a permittivity scalar with a positive real part in the near-infrared regime, and ICMs that do not involve metallic components are generally characterized by permittivity scalars with positive real parts.

Figure 1 shows the real and imaginary parts of ε^{Br} , ξ^{Br} , and μ^{Br} as functions of f_a , when $\varepsilon^a = (2.0 - 0.02i)\varepsilon_0$, $\varepsilon^b = (3 + 0.01i)\varepsilon_0$, $\xi^b = (0.1 + 0.001i)/c_0$, and $\mu^b = (0.95 + 0.0002i)\mu_0$, with ε_0 and μ_0 being the permittivity and permeability of free space, respectively, and $c_0 = 1/\sqrt{\varepsilon_0\mu_0}$. The component material ‘b’ is guaranteed to be dissipative since $[\text{Im}(\xi^b)]^2 < \text{Im}(\varepsilon^b)\text{Im}(\mu^b)$ [23]. The chosen values of ε^a , ε^b , ξ^b , and μ^b are physically plausible [6, 24, 25].

The real and imaginary parts of ε^{Br} , ξ^{Br} , and μ^{Br} vary almost linearly in figure 1 as f_a increases from 0 to 1, with

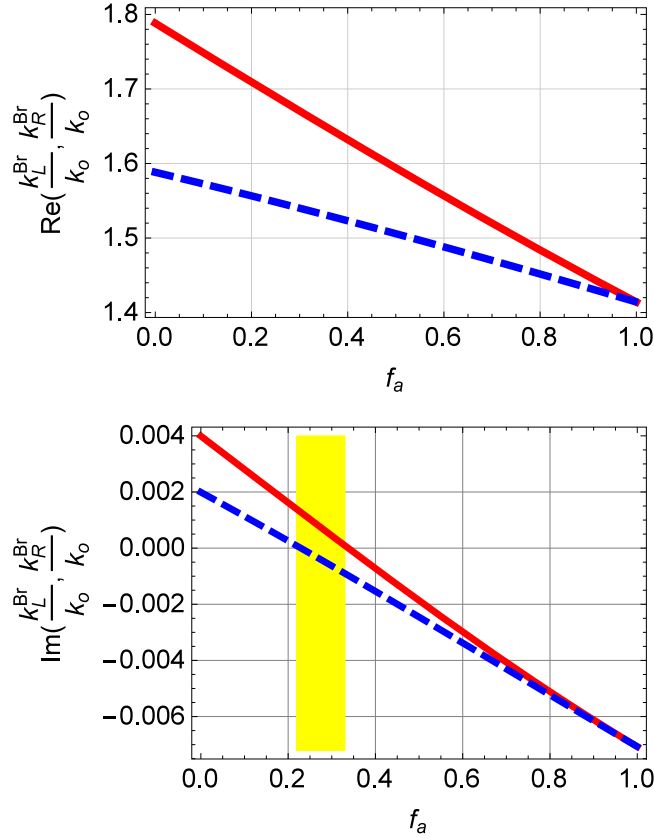


Figure 2. The real and imaginary parts of the relative wavenumbers k_L^{Br} (red solid curves) and k_R^{Br} (blue dashed curves) in the HCM, normalized with respect to the free-space wavenumber $k_0 = \omega/c_0$, plotted against volume fraction f_a . The component materials are the same as for figure 1. The volume fraction range where $\text{Im}(k_L^{\text{Br}})\text{Im}(k_R^{\text{Br}}) < 0$ is shaded in yellow.

their endpoints complying with the limits

$$\varepsilon^{\text{Br}} \rightarrow \begin{cases} \varepsilon^a \\ \varepsilon^b \end{cases}, \quad \xi^{\text{Br}} \rightarrow \begin{cases} 0 \\ \xi^b \end{cases}, \quad \mu^{\text{Br}} \rightarrow \begin{cases} \mu_0 \\ \mu^b \end{cases} \quad \text{as } f_a \rightarrow \begin{cases} 1 \\ 0 \end{cases}. \quad (5)$$

In particular, the sign of $\text{Im}(\varepsilon^{\text{Br}})$ changes at $f_a \approx 0.27$.

The real and imaginary parts of the wavenumbers for the HCM—namely k_L^{Br} and k_R^{Br} per equation (4) with the superscript ‘Br’ attached to k_L , k_R , ε , ξ , and μ therein—are plotted against volume fraction f_a in figure 2. Whereas $\text{Re}(k_L^{\text{Br}}) > 0$ and $\text{Re}(k_R^{\text{Br}}) > 0$ for any $f_a \in [0, 1]$, three mutually disjoint f_a -regimes can be identified for the signs of $\text{Im}(k_L^{\text{Br}})$ and $\text{Im}(k_R^{\text{Br}})$ as follows:

- (i) $f_a \in [0, 0.22)$, when $\text{Im}(k_L^{\text{Br}}) > 0$ and $\text{Im}(k_R^{\text{Br}}) > 0$;
- (ii) $f_a \in (0.22, 0.33)$, when $\text{Im}(k_L^{\text{Br}}) > 0$ and $\text{Im}(k_R^{\text{Br}}) < 0$; and
- (iii) $f_a \in (0.33, 1]$, when $\text{Im}(k_L^{\text{Br}}) < 0$ and $\text{Im}(k_R^{\text{Br}}) < 0$.

Clearly then, a continuous range of values of the volume fraction f_a , specifically $f_a \in (0.22, 0.33)$ for the example presented in figure 2, can exist wherein \mathbf{Q}_L attenuates whereas \mathbf{Q}_R amplifies.

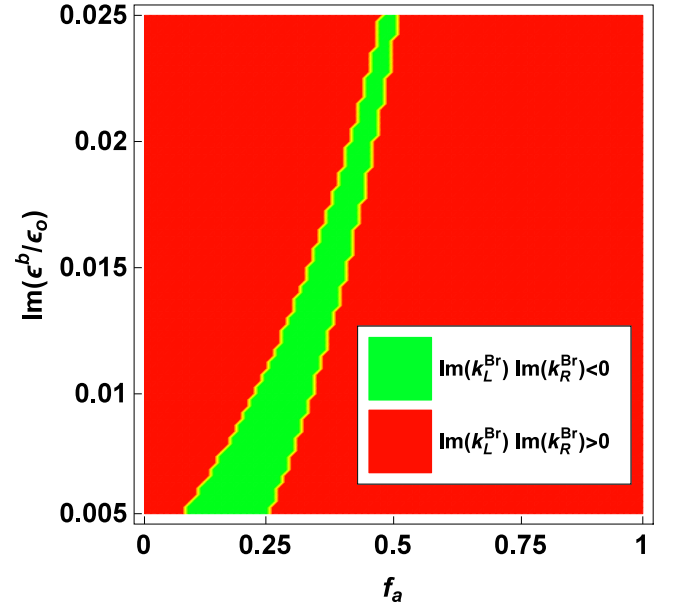


Figure 3. Regions in the parameter space of $\text{Im}(\varepsilon^b/\varepsilon_0)$ and f_a where $\text{Im}(k_L^{\text{Br}})\text{Im}(k_R^{\text{Br}}) < 0$ (green) and $\text{Im}(k_L^{\text{Br}})\text{Im}(k_R^{\text{Br}}) > 0$ (red). The component materials are the same as for figure 1, except that $\text{Im}(\varepsilon^b/\varepsilon_0) \in (0.005, 0.025)$.

3.2. Dissipation

The ICM used as the component material ‘b’ for the numerical example considered in figures 1 and 2 exhibits weak dissipation. Can simultaneous amplification and attenuation still be achieved if the component material ‘b’ is rather more dissipative? We address this question next.

In figure 3, the regions in the parameter space of $\text{Im}(\varepsilon^b)$ and f_a where $\text{Im}(k_L^{\text{Br}})\text{Im}(k_R^{\text{Br}}) < 0$ (green) and $\text{Im}(k_L^{\text{Br}})\text{Im}(k_R^{\text{Br}}) > 0$ (red) are delineated. The component materials are the same as for figures 1 and 2, except that $\text{Im}(\varepsilon^b/\varepsilon_0) \in (0.005, 0.025)$. Clearly, simultaneous amplification and attenuation is achieved for all values of $\text{Im}(\varepsilon^b)$ considered, and the range of f_a over which $\text{Im}(k_L^{\text{Br}})\text{Im}(k_R^{\text{Br}}) < 0$ decreases as $\text{Im}(\varepsilon^b)$ increases. Parenthetically, $\text{Re}(k_L^{\text{Br}}) > 0$ and $\text{Re}(k_R^{\text{Br}}) > 0$ for all results represented in figure 3.

In view of figure 3, the question arises: if $\text{Im}(\varepsilon^b)$ becomes much larger (i.e., sufficiently large that the component material ‘b’ may be regarded as a strongly dissipative ICM), can simultaneous amplification and attenuation still be achieved? The answer to this question may be inferred from figure 4, wherein the real and imaginary parts of k_L^{Br} and k_R^{Br} are plotted against f_a , as in figure 2 but with $\varepsilon^a = (2 - 0.1i)\varepsilon_0$ and $\varepsilon^b = (3 + i)\varepsilon_0$. We see that $\text{Im}(k_L^{\text{Br}})\text{Im}(k_R^{\text{Br}}) < 0$, and $\text{Re}(k_L^{\text{Br}}) > 0$ and $\text{Re}(k_R^{\text{Br}}) > 0$, for the volume fraction range $f_a \in (0.872, 0.887)$. Therefore, we infer that a strongly dissipative component material ‘b’ is no barrier to achieving simultaneous amplification and attenuation, albeit the f_a range over which simultaneous amplification and attenuation is achieved is very small when $\text{Im}(\varepsilon^b)$ is large.

Let us next explore the ranges of values of $\text{Im}(\varepsilon^a)$ and f_a for which simultaneous amplification and attenuation is possible. In figure 5, the regions in the parameter space of

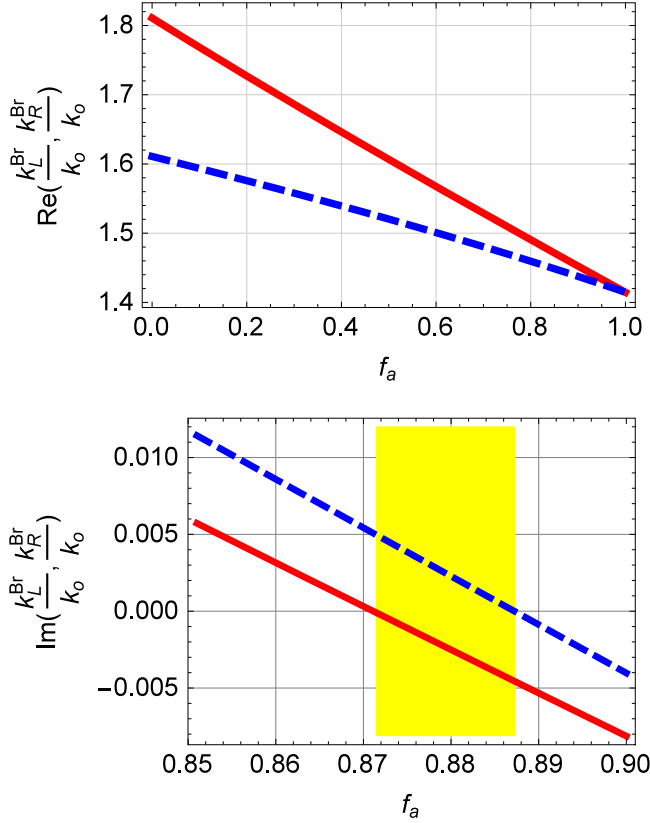


Figure 4. As figure 2 but with $\varepsilon^a = (2 - 0.1i)\varepsilon_0$ and $\varepsilon^b = (3 + i)\varepsilon_0$.

$\text{Im}(\varepsilon^a)$ and f_a where $\text{Im}(k_L^{\text{Br}})\text{Im}(k_R^{\text{Br}}) < 0$ (green) and $\text{Im}(k_L^{\text{Br}})\text{Im}(k_R^{\text{Br}}) > 0$ (red) are delineated. The component materials are the same as for figures 1 and 2, except that $\text{Im}(-\varepsilon^a/\varepsilon_0) \in (0.005, 0.045)$. For all values of $\text{Im}(\varepsilon^a)$ considered, simultaneous amplification and attenuation is achieved, and the size of the range of f_a over which $\text{Im}(k_L^{\text{Br}})\text{Im}(k_R^{\text{Br}}) < 0$ remains approximately the same as $\text{Im}(\varepsilon^a)$ increases. Parenthetically, $\text{Re}(k_L^{\text{Br}}) > 0$ and $\text{Re}(k_R^{\text{Br}}) > 0$ for all results represented in figure 5.

3.3. Chirality parameter

If ξ is replaced by $-\xi$ in equation (4), then k_L and k_R in equation (4) are interchanged. Also, if the sign of $\text{Re}(\xi)$ is reversed in equation (4), then $\text{Re}(k_L)$ and $\text{Re}(k_R)$ are interchanged, but $\text{Im}(k_L)$ and $\text{Im}(k_R)$ remain unchanged. Therefore, if ξ^b were to be replaced by $-\xi^b$ for the homogenization scenario represented in figures 1 and 2, then the handedness of the Beltrami field that is amplified/attenuated for the regime wherein $\text{Im}(k_L^{\text{Br}})\text{Im}(k_R^{\text{Br}}) < 0$ will be reversed. To illustrate this point, in figure 6 the wavenumbers k_L^{Br} and k_R^{Br} are plotted against f_a for the same homogenization scenario as represented in figures 1 and 2 but with $\xi^b = -(0.1 + 0.001i)/c_0$. For $f_a \in (0.22, 0.33)$, we infer from figure 6 that \mathbf{Q}_L is amplified and \mathbf{Q}_R is attenuated, whereas from figure 2 it is \mathbf{Q}_R that is amplified and \mathbf{Q}_L that is attenuated.

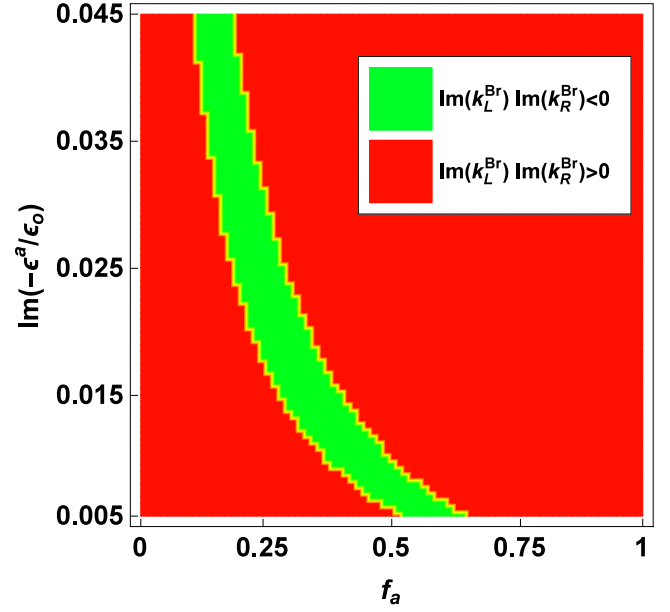


Figure 5. Regions in the parameter space of $\text{Im}(-\varepsilon^a/\varepsilon_0)$ and f_a where $\text{Im}(k_L^{\text{Br}})\text{Im}(k_R^{\text{Br}}) < 0$ (green) and $\text{Im}(k_L^{\text{Br}})\text{Im}(k_R^{\text{Br}}) > 0$ (red). The component materials are the same as for figure 1, except that $\text{Im}(-\varepsilon^a/\varepsilon_0) \in (0.005, 0.045)$.

The magnitude of the chirality parameter used for the component material ‘b’ in the calculations represented in figures 1–6 is consistent with that associated with certain isotropic chiral metamaterials [14], but is considerably larger than the magnitude generally associated with organic ICMs [7, 26], for example. Therefore the question arises: Can simultaneous amplification and attenuation still be achieved for much smaller magnitudes of ξ^b , within the range associated with organic ICMs [12], for example? This question is addressed via figure 7, wherein the real and imaginary parts of k_L^{Br} and k_R^{Br} are plotted against f_a , as in figure 2 but with $\xi^b = (0.1 + 0.001i) \times 10^{-3}/c_0$. The real parts of k_L^{Br} and k_R^{Br} are indistinguishable to the naked eye in figure 2, but are positive-valued for $f_a \in (0, 1)$. For the volume fraction range $f_a \in (0.29120, 0.29131)$ we see that $\text{Im}(k_L^{\text{Br}})\text{Im}(k_R^{\text{Br}}) < 0$. Therefore, simultaneous amplification and attenuation is achievable for $\xi^b = (0.1 + 0.001i) \times 10^{-3}/c_0$, albeit for a tiny range of the volume fraction of the active component material.

4. Closing remarks

In the foregoing analysis, a physically plausible means of achieving an ICM for which left-handed Beltrami fields are amplified whereas right-handed Beltrami fields are attenuated (or vice versa) is conceptualized. Specifically, for an ICM arising from the homogenization of a dissipative ICM and an active dielectric material, our numerical results reveal that:

- The HCM exhibits simultaneous amplification and attenuation across a relatively wide range of the volume fraction of the active component material, provided that

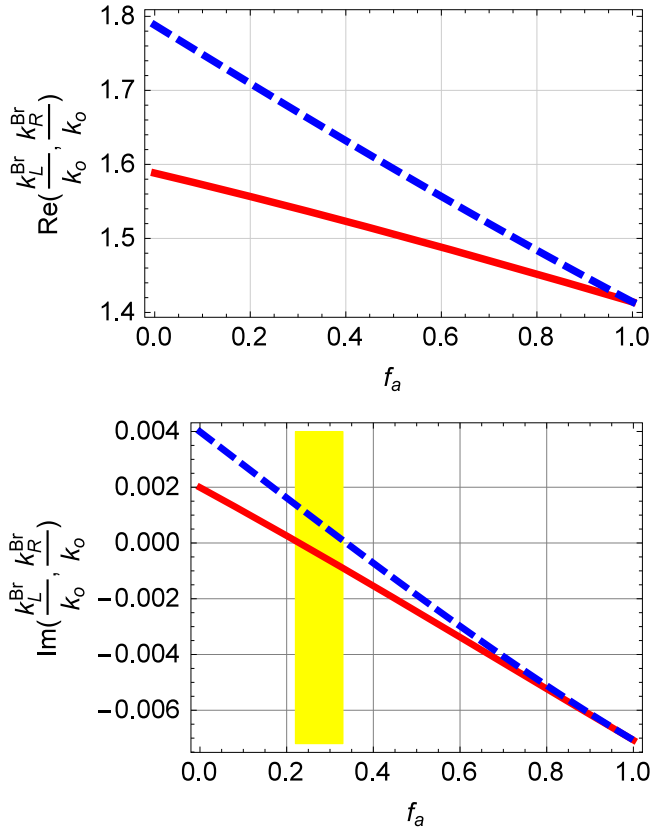


Figure 6. As figure 2 but with ζ^b replaced by $-\zeta^b$.

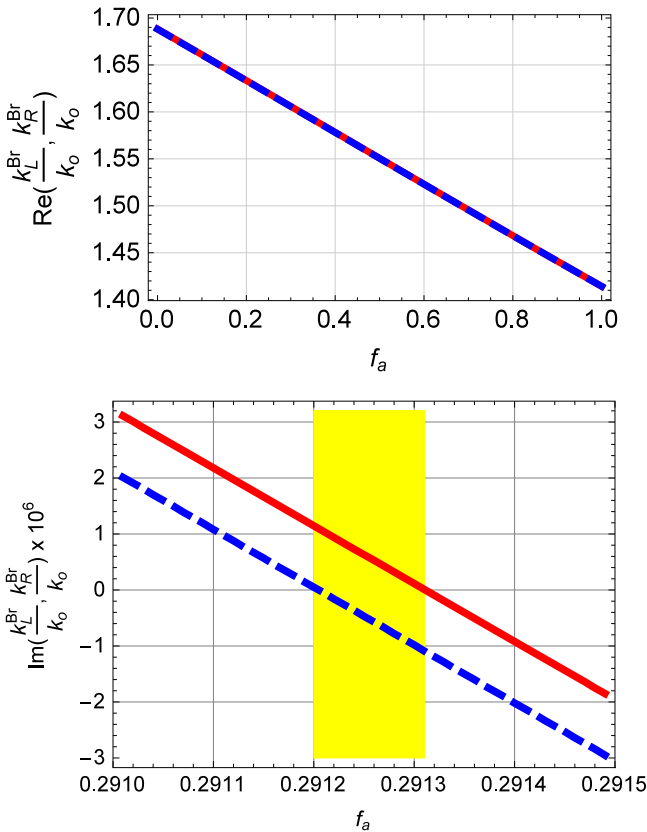


Figure 7. As figure 2 but with $\zeta^b c_0 = (0.1 + 0.001i) \times 10^{-3}$.

the ICM component exhibits relatively low dissipation and has a relatively large chirality parameter.

- (b) The range of volume fraction of the active component material for which simultaneous amplification and attenuation is allowed decreases as the ICM component becomes more dissipative and its chirality parameter becomes smaller in magnitude.
- (c) When the ICM component exhibits relatively high dissipation or has a chirality parameter of relatively small magnitude, simultaneous amplification and attenuation can still be achieved albeit across a very small range of volume fraction of the active component material.
- (d) If the chirality parameter of the ICM component changes sign for an HCM that exhibits simultaneous amplification and attenuation, then the handedness of the Beltrami field that is amplified/attenuated swaps over.

These results open the door for circular polarizers of a new type.

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